Reconnection rate for the steady-state Petschek model

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Abstract

Reconnection rate is found for the canonical simplest case of steady-state two-dimensional symmetric reconnection in an incompressible plasma by matching of outer Petschek solution and internal diffusion region solution. The reconnection rate obtained naturally incorporates both Sweet–Parker and Petschek regimes, the latter seems to be possible only for the case with strongly localized resistivity.

I. INTRODUCTION

Magnetic reconnection is an energy conversion process which occurs in astrophysical, solar, space and laboratory plasmas (e.g., Hones[1]; Priest[2]). First attempts to explain the fast energy release in solar flares based on pure resistive magnetic field dissipation (Sweet[3]; Parker[4]) showed that the energy conversion rate is estimated as $1/\sqrt{Re_m}$, where

$$Re_m = \frac{V_A L}{\eta} \tag{1}$$

is the global Reynolds number, L is the half-length of reconnection layer, V_A is Alfvénic velocity, and η is resistivity. For typical conditions in the solar corona the Sweet-Parker rate turns out to be orders of magnitudes too small when compared to experimental data.

In 1964 Petschek[5] pointed out that in a highly–conducting plasma dissipation needs only to be present within a small region known as the diffusion region, and energy conversion occurs primarily across non–linear waves, or shocks. This gives another estimation of the maximum reconnection rate $1/\ln Re_m$ which is much more favorable for energy conversion.

Unfortunately, up to the present it is still unclear which conditions make Petschek-type reconnection to be possible and which are responsible for the Sweet-Parker regime. The fact is that numerical simulations (Biskamp,1986, Scholer, 1989) were not able to reproduce solution of Petschek type but rather were in favor of Sweet-Parker solution unless the resistivity was localized in a small region (e.g., Scholer 1989, Yan, Lee and Priest, 1992, Ugai, 1999). The laboratory experiments also seem to observe Sweet-Parker regime of reconnection (Uzdensky et al., 1996, Ji et al.,1999).

From the mathematical point of view the problem of reconnection rate is connected with the matching of a solution for the diffusion region where dissipation is important, and solution for the convective zone where ideal MHD equations can be used. But up to now this question is still not resolved even for the canonical simplest case of steady-state two-dimensional symmetric reconnection in an incompressible plasma.

It is the aim of this paper to present a matching procedure for the canonical reconnection problem. The reconnection rate obtained from the matching turns out to incorporate naturally both Petschek and Sweet-Parker regimes as limiting cases.

Petschek solution

We consider the simplest theoretical system consisting of a two–dimensional current sheet which separates two uniform and identical plasmas with oppositely oriented magnetic fields $\pm\mathbf{B_0}$. Petschek (1964) pointed out that the diffusion region can be considerably smaller than the whole size of the reconnection layer and that

the outer region contain two pairs of standing slow shocks. These shocks deflect and accelerate the incoming plasma from the inflow region into two exit jets wedged between the shocks (see Figure 1). This jet area between the shocks with accelerated plasma is traditionally called outflow region.

In the dimensionless form the Petschek solution can be presented as follows (Petschek, 1964, for details see Vasyliunas, 1975):

Inflow region:

$$v_x = 0;\ v_y = -\varepsilon, \tag{2}$$

$$B_x = 1 - \frac{4\varepsilon}{\pi} \ln \frac{1}{\sqrt{x^2+y^2}};\quad B_y = \frac{4\varepsilon}{\pi} \arctan \frac{x}{y}. \tag{3}$$

Outflow region:

$$v_x = 1;\ v_y = 0;\ B_x = 0;\ B_y = \varepsilon. \tag{4}$$

Equation of shock in the first quadrant is the following:

$$y = \varepsilon x. \tag{5}$$

Here x, y are directed along the current sheet and in the perpendicular direction, respectively. We normalized the magnetic field to B_0, length to L, plasma velocity to Alfvénic velocity V_A, and electric field E to Alfvénic electric field $E_A = V_A B_0$.

The reconnection rate

$$\varepsilon = E/E_A << 1 \tag{6}$$

is supposed to be a small parameter of the problem.

Expressions (2-5) are the asymptotic solution with respect to ε of the MHD system of equations

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\mathbf{P} + (\mathbf{B}\cdot\nabla)\mathbf{B}, \tag{7}$$

$$\mathbf{E} + (\mathbf{v}\times\mathbf{B}) = \tfrac{1}{Re_m}\mathrm{curl}\mathbf{B}, \tag{8}$$

$$\mathrm{div}\mathbf{B} = 0,\ \mathrm{div}\mathbf{v} = 0, \tag{9}$$

and the Rankine-Hugoniot shock relations in the limit $Re_m \to \infty$. Petschek did not obtain a solution in the diffusion region, instead he estimated maximum reconnection rate as $1/\ln Re_m$ of using some simple physical suggestion . Generally speaking, this implies that the Petschek model gives any reconnection rate from Sweet-Parker value $1/\sqrt{Re_m}$ up to $1/\ln Re_m$, and it is still unclear whether Petschek reconnection faster than Sweet-Parker reconnection is possible. The problem can be solved by matching of a solution for the diffusion region and Petschek solution (2-5).

Diffusion region scaling

We renormalize the MHD equations to the new scales $B_0',\ V_A',\ E_A' = B_0' V_A'$, where all quantities are supposed to be taken at the diffusion region upper boundary, and

at the half length of the diffusion region l_d. We have to use the dissipative MHD equations (7–9) for the diffusion region with Reynolds number

$$Re'_m = \frac{V'_A l_d}{\eta}, \tag{10}$$

and electric field $E = \varepsilon'$.

The scaling for the diffusion region is similar to that for the Prandtl viscous layer (see Landau and Lifshitz, 1985):

$$\begin{gathered} x',\ B'_x,\ v'_x,\ P' \sim O(1), \\ y',\ B'_y,\ v'_y,\ \varepsilon' \sim 1/\sqrt{Re'_m}. \end{gathered} \tag{11}$$

Consequently, the new boundary layer variables are the following:

$$\begin{gathered} \tilde{x} = x',\ \tilde{B}_x = B'_x,\ \tilde{v}_x = v'_x,\ \tilde{P} = P', \\ \tilde{y} = y'\sqrt{Re'_m},\ \tilde{B}_y = B'_y\sqrt{Re'_m},\ \tilde{v}_y = v'_y\sqrt{Re'_m},\ \tilde{\varepsilon} = \varepsilon'\sqrt{Re'_m}. \end{gathered} \tag{12}$$

The diffusion region Reynolds number is supposed to be $Re'_m >> 1$, and therefore in the zero-order with respect to the parameter $1/\sqrt{Re'_m}$ the boundary layer equations turn out to be:

$$\tilde{v}_x \tfrac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \tfrac{\partial \tilde{v}_x}{\partial \tilde{y}} - \tilde{B}_x \tfrac{\partial \tilde{B}_x}{\partial \tilde{x}} - \tilde{B}_y \tfrac{\partial \tilde{B}_x}{\partial \tilde{y}} = -\tfrac{\partial \tilde{P}(\tilde{x})}{\partial \tilde{x}}, \tag{13}$$

$$\mathrm{div}\tilde{\mathbf{B}} = 0,\ \mathrm{div}\tilde{\mathbf{v}} = 0, \tag{14}$$

$$\tilde{P} = \tilde{P}(\tilde{x}), \tag{15}$$

$$\tilde{v}_y \tilde{B}_x - \tilde{v}_x \tilde{B}_y - \tilde{\varepsilon} = \tilde{\eta}(\tilde{x},\tilde{y}) \tfrac{\partial \tilde{B}_x}{\partial \tilde{y}}, \tag{16}$$

where $\tilde{\eta}(\tilde{x},\tilde{y})$ is the normalized resistivity of the plasma with maximum value 1.

Unfortunately, the appropriate exact solutions of the boundary layer equations (13–16) are unknown, therefore we have to solve the problem numerically. The main difficulty is that the internal reconnection rate $\tilde{\varepsilon}$ is unknown in advance and has to be determined for given resistivity $\tilde{\eta}(\tilde{x},\tilde{y})$, given total pressure $\tilde{P}(\tilde{x})$, and $\tilde{B}_x(\tilde{x})$ given at the upper boundary of the diffusion region. In addition, the solution must have Petschek-type asymptotic behaviour (2–5) outside of the diffusion region.

Although we are looking for a steady-state solution, from the simulation point of view it is advantageous to use relaxation method and solve numerically the following unstationary system of boundary layer MHD equations:

$$\tfrac{\partial \tilde{v}}{\partial t} + \tilde{v}_x \tfrac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \tfrac{\partial \tilde{v}_x}{\partial \tilde{y}} - \tilde{B}_x \tfrac{\partial \tilde{B}_x}{\partial \tilde{x}} - \tilde{B}_y \tfrac{\partial \tilde{B}_x}{\partial \tilde{y}} = -\tfrac{\partial \tilde{P}(\tilde{x})}{\partial \tilde{x}}, \tag{17}$$

$$\tfrac{\partial \tilde{\mathbf{B}}}{\partial t} = \mathrm{curl}(\tilde{\mathbf{v}} \times \tilde{\mathbf{B}}) - \mathrm{curl}\left(\eta(\tilde{x},\tilde{y})\ \mathrm{curl}\tilde{\mathbf{B}}\right), \tag{18}$$

$$\mathrm{div}\tilde{\mathbf{B}} = 0,\ \mathrm{div}\tilde{\mathbf{v}} = 0. \tag{19}$$

Starting with an initial MHD configuration under fixed boundary conditions we look for convergence of the time-dependent solutions to a steady state.

As initial configuration we choose a X-type flow and magnetic field: $\tilde{v}_x = \tilde{x},\ \ \tilde{v}_y = -\tilde{y},\ \ \tilde{B}_x = \tilde{y},\ \ \tilde{B}_y = -\tilde{x}$. The distribution of the resistivity is traditional (see (Ugai,1999, Scholer 1985)):

$$\eta(\tilde{x}, \tilde{y}) = de^{(-s_x\tilde{x}^2 - s_y\tilde{y}^2)} + f, \tag{20}$$

with $d + f = 1$ where coefficient d describes inhomogeneous resistivity, and f is responsible for the background resistivity.

The problem under consideration consists essentially of two coupled physical processes: diffusion and wave propagation. To model these processes, two-step with respect to time numerical scheme has been used. At first, convectional terms were calculated using the Godunov characteristic method, and then the elliptical part was treated implicitly.

Calculations were carried out on a rectangular uniform grid 100×145 in the first quadrant with the following boundary conditions:
Lower boundary: symmetry conditions $\partial\tilde{v}_x/\partial y = 0,\ \ \tilde{v}_y = 0,\ \ B_x = 0$; induction equation (18) has been used to compute the B_y component at the x–axis.
Left boundary: symmetry conditions $\tilde{v}_x = 0,\ \ \partial\tilde{v}_y/\partial x = 0,\ \ \partial\tilde{B}_x/\partial x = 0,\ \ \tilde{B}_y = 0$.
Right boundary: free conditions $\partial\tilde{v}_x/\partial x = 0,\ \ \partial\tilde{v}_y/\partial x = 0$.
Upper (inflow) boundary: $\tilde{v}_x = 0,\ \ \tilde{B}_x = 1$.

Note, that this implies that we do not prescribe the incoming velocity, and hence the reconnection rate: the system itself has to determine how fast it wants to reconnect.

The total pressure can be fixed to 1 in the zero-order approximation: $\tilde{P} = 1$.

Let us discuss the result of our simulations. For the case of localized resistivity where we chose $d = 0.95,\ \ f = 0.05,\ \ s_x = s_y = 1$ in the equation (20), the system reaches Petschek steady state (see Figure 2) with clear asymptotic behaviour, pronounced slow shock, and the reconnection rate turns out to be $\tilde{\varepsilon} \sim 0.7$.

From the other hand, for the case of homogeneous resistivity $d = 0,\ \ f = 1$, the system reaches Sweet-Parker state (see Figure 3) with much less reconnection rate $\tilde{\varepsilon} \sim .25$ even if the Petschek solution has been used as initial configuration (see also (Ugai, 1999, Scholer,1989)). This seems to imply that Petschek-type reconnection is possible only if the resistivity of the plasma is localized in a small region, and for constant resistivity the Sweet-Parker regime is realized.

The size of the diffusion region l_d can be defined as the size of the region where the convective electric field $E = v \times B$ (which is zero at the origin) reaches the asymptotic value $\tilde{\varepsilon}$ (or, some level, say $0.95\tilde{\varepsilon}$). For the case of localized resistivity l_d practically coincides with the scale of the inhomogeneity of the conductivity. In principal, there might be a possibility to produce Petschek-type reconnection with constant resistivity using a highly inhomogeneous behaviour of the MHD parameters

at the upper boundary (narrow stream, for example, see Chen et al.,1999), and then l_d has the meaning of the scale of this shearing flow or other boundary factor which causes the reconnection.

Matching procedure

We have only a numerical solution for the diffusion region, and this makes it difficult for the matching procedure because the latter needs an analytical presentation of the solutions to be matched. The only way out left is to continue the diffusion region solution to the inflow region using dates known from the simulation distribution of the B_y component along the upper boundary of the diffusion region. Then try to match the solutions in the current free inflow region at the distance $r \sim l_d$ (see Figure 1).

As can be seen from equation (3) the B_x component of the Petschek solution diverges at the origin $B_x \to -\infty$ when $r = \sqrt{x^2 + y^2} \to 0$. This singularity is a consequence of the fact that dissipation actually has not been taken into account for the solution (2- 5) which is nevertheless still valid untill the distances of the order of the size of diffusion region is l_d.

In order to be adjusted to the Petschek solution, the B'_y component must have the following limit for $x/l_d \to \infty$ at the upper boundary of the diffusion region :

$$B'_y(x/l_d) \to 2\varepsilon. \tag{21}$$

We can obtain the asymptotic behaviour of B'_x for $r > l_d$ region using a Poisson-like integral presentation:

$$\begin{aligned} B'_x(x', y') = B'_0 + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\partial B'^{(1)}_y(\tilde{x}, 0)}{\partial x} \ln \frac{\sqrt{(x - \tilde{x})^2 + y^2}}{l_d} d\tilde{x} = \\ B'_0 + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\partial B'^{(1)}_y(\xi, 0)}{\partial \xi} \left\{ \ln \frac{\sqrt{x^2 + y^2}}{l_d} + \frac{\xi^2 - 2x\xi}{x^2 + y^2} \right\} d\xi = \\ B'_0 + \frac{4\varepsilon}{\pi} \ln \frac{r}{l_d} + O(1/r), \end{aligned} \tag{22}$$

where $\xi = x/l_d$. This gives an outer expansion for the inner solution. On the other hand a convective solution (3) can be rewritten in the following form in order to determine the inner expansion of the outer solution:

$$B_x = 1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{r} = 1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{l_d} - \frac{4\varepsilon}{\pi} \ln \frac{l_d}{r}. \tag{23}$$

Equating these two asymptotic expansions we obtain the matching relation:

$$B'_0 = 1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{l_d}, \tag{24}$$

Now everything is ready to determine the reconnection rate. The electric field must be constant in the whole inflow region, hence

$$v' B_0' = v B_0, \tag{25}$$
$$\varepsilon' B_0'^2 = \varepsilon B_0^2, \tag{26}$$

where the definition of the reconnection rates $\varepsilon' = v'/B_0', \;\; \varepsilon = v/B_0$ has been used. Bearing in mind that $\varepsilon' = \tilde{\varepsilon}/\sqrt{R} e_m'$ (see scaling (12) we obtain:

$$\tilde{\varepsilon} B_0'^{\,3/2} = \varepsilon B_0^{3/2} \sqrt{\frac{l_d B_0}{\eta}}. \tag{27}$$

Substituting B_0' from the equation (24) we determine finally the following equation for the reconnection rate ε :

$$\tilde{\varepsilon}(1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{l_d})^{3/2} = \varepsilon \sqrt{Re_m \frac{l_d}{L}}, \tag{28}$$

where Re_m is the global Reynolds number (1), and the internal reconnection rate $\tilde{\varepsilon}$ has to be found from the simulation of the diffusion region problem.

For small ε there is an analytical expression:

$$\varepsilon = \frac{\tilde{\varepsilon}}{\sqrt{Re_m \frac{l_d}{L}} + \frac{6}{\pi} \tilde{\varepsilon} \ln \frac{L}{l_d}}. \tag{29}$$

Here $\tilde{\varepsilon}$ is an internal reconnection rate determined from the numerical solution: $\tilde{\varepsilon} \sim 0.7$.

Discussion and conclusion

Equations (28,29) give the unique reconnection rate for known parameters of the current sheet $L, \;\; B_0, \;\; V_A, \;\; \eta, \;\; l_d$. For sufficiently long diffusion region such that $\sqrt{Re_m \frac{l_d}{L}} >> \frac{6}{\pi}\tilde{\varepsilon} \ln \frac{L}{l_d}$, the equation (29) corresponds to Sweet-Parker regime $\varepsilon \sim \tilde{\varepsilon}/\sqrt{Re_m \frac{l_d}{L}}$. For the opposite case of resistivity constrained in a small region $\varepsilon \sim \frac{\pi}{6}/\ln \frac{L}{l_d}$ we have Petschek reconnection. Hence, reconnection rate (28,29) naturally incorporates both regimes obtained in simulations (Scholer, Ugai, Biskump).

We were not able to reproduce Petschek regime using variation of MHD parameters at the upper boundary with homogeneous resistivity, a probably solution (Chen, 1999) of this problem either is essentially time-dependent or corresponds to the case of strong reconnection. According to our simulations, for Petschek state to exist a strongly localized resistivity is needed, and for the spatially homogeneous resistivity $l_d = L$ Sweet–Parker regime seems to be always the case. This result resolves old

question about conditions which are necessary for Petschek-type reconnection to appear.

It is interesting that for the deriving of equations (28,29) the only value which has been actually used is the internal reconnection rate $\tilde{\varepsilon}$ obtained from the numerical solution, but the distribution of the B_y component along the upper boundary of the diffusion region does not contribute at all (besides asymptotic behaviour (22)) in the zero–order approximation considered above. Of course, from the mathematical point of view it is important that diffusion region solution exists and has Petschek–like asymptotic behaviour (2–4).

The strongly localized resistivity is often the relevant case in space plasma applications, but for the laboratory experiments where the size of a device is relatively small the Sweet–Parker regime is expected.

VIII. ACKNOWLEDGEMENTS

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IX. REFERENCES

E. W. Hones, Jr., *Magnetic Reconnection in Space and Laboratory Plasmas* (Geophysical Monograph 30, AGU, Washington, 1984).

E. R. Priest, Rep. Progr. Phys., **48**, 955 (1985).

P. A. Sweet, in *Electromagnetic Phenomena in Cosmic Physics*, edited by B. Lehnert (Cambridge University Press, London, 1958), p. 123.

E. N. Parker, Astrophys. J. Suppl. Ser., **8**, 177 (1963).

H. E. Petschek, in *AAS–NASA Symposium of the Physics of Solar Flares, NASA–SP 50*, edited by W. N. Hess (National Aeronautics and Space Administration, Washington, DC, 1964), p. 425.

V. M. Vasyliunas, Rev. Geophys. Space Phys., **13**, 303 (1975).

D. Biskamp, Magnetic reconnection via current sheets, Phys. Fluids, 29, 1520, 1986.

Scholer, M., Undriven reconnection in an isolated current sheet, J.Geophys. Res.,94, 8805, 1989.

Yan, M., L.C.Lee and E.R.Priest, Fast magnetic reconnection with small shock angles, J.geophys.Res., 97, 8277, 1992.

Ugai,M., Computer studies on the spontaneous fast reconnection model as a nonlinear instability, Phys. Plasmas,6, 1522, 1999.

Uzdensky D.A., R.M.Kulsrud, and M. Yamada, Phys.Plasmas, 3, 1220, 1996.

L. D. Landau and E. M. Lifschitz, *Klassische Feldtheorie* (Akademie–Verlag, Berlin, 1984).

Chen, T., Z. X. Liu, and X. X. Zhang, Transient reconnection caused by the impact and switch-off of a transverse shear flow, Phys. Plasmas, 6, 2393, 1999.

Ji, H., M. Yamada, S. Hsu, R. Kulsrud, T. Carter, and S. Zaharia, Magnetic reconnection with Sweet–Parker characteristics in two-dimensional laboratory plasmas, Phys. Plasmas, 6, 1743, 1999.

Figure Captions

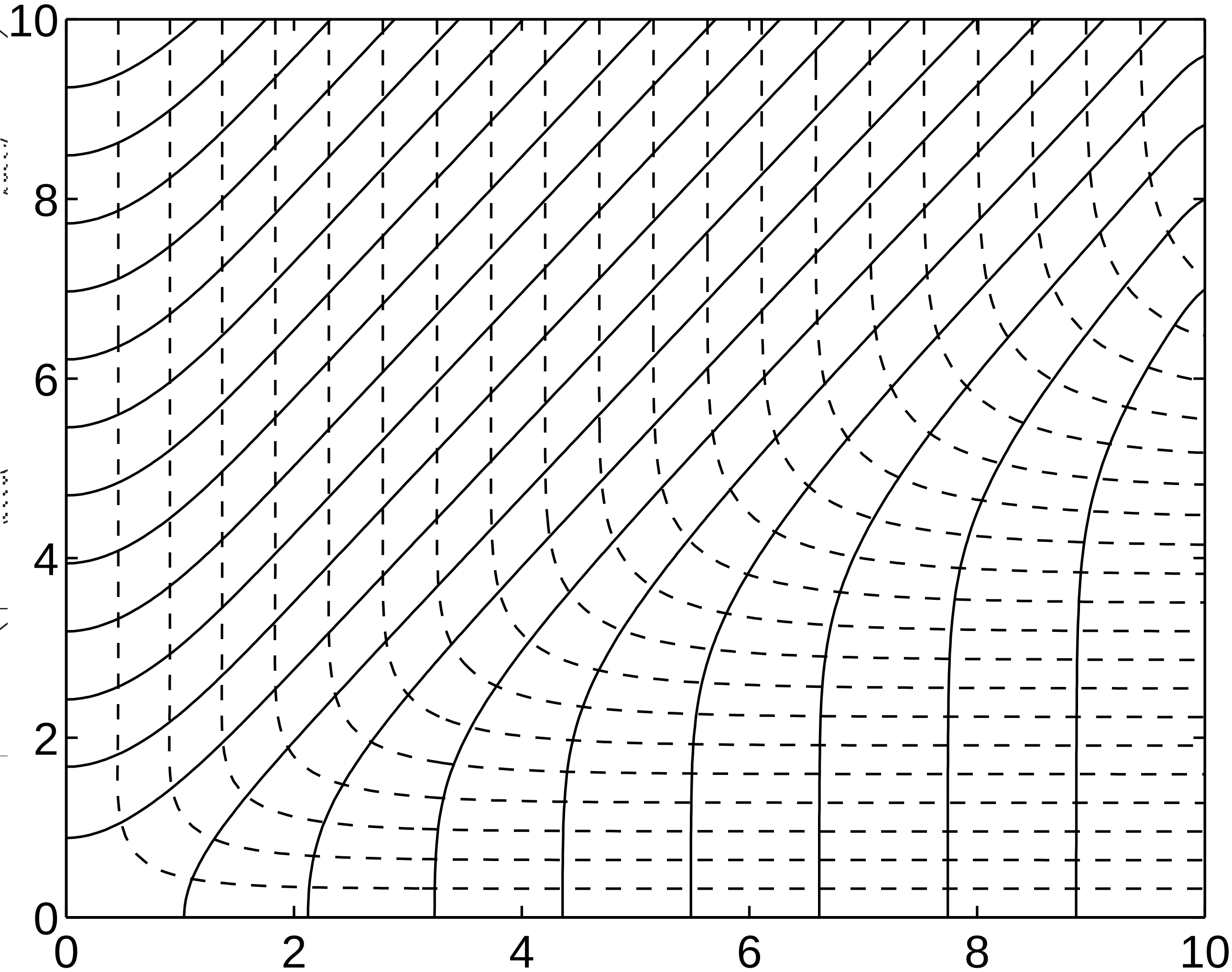
Figure 1: Scheme of matching of the outer Petschek solution and diffusion region solution.

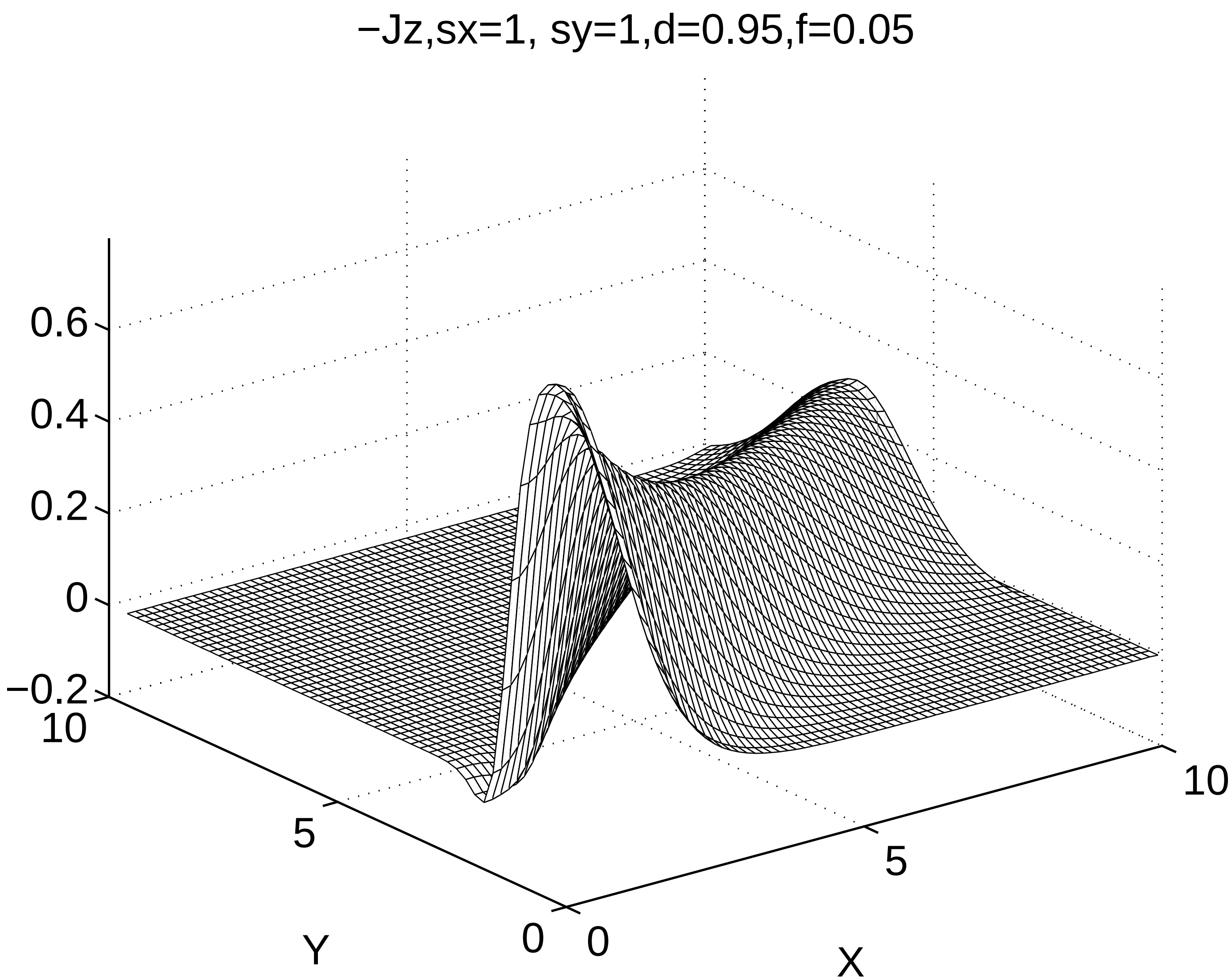
Figure 2: Configuration of magnetic field lines (solid line) and stream lines (dashed line) for the numerical simulation of the diffusion region.

Figure 3: Three-dimensional plot of current density shows Petschek shock

Inflow region

Petschek solution

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0.2
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10
X